AXISYMMETRIC MACRODISPERSION PROBLEMS IN STRATIFIED POROUS MEDIA

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The mass transfer of impurities in stratified porous media can be described by the diffusion equation with a convective term. In this case the dispersion coefficient is proportional to the square of the filtration velocity. In this paper we study the problem of the infusion of a mixture into a stratum through a hole and the subsequent pumping of it out through the hole, and we give the solution of the problem for various boundary conditions and their asymptotic representations. From the results it is possible to construct a method of testing using a single-hole method in order to determine the parameters of transport.

The mass transport of dynamically neutral impurities in stratified porous media when the mixtures move in a direction parallel to the stratification is characterized by the phenomenon of longitudinal macrodispersion [1-3] In its asymptotic formulation the process is described by the diffusion equation with a convective term. The total dispersion coefficient D*, taking molecular diffusion, convective diffusion, and macrodispersion into account, then has the form

$$D^* = D + \lambda v + \delta v^2 \tag{0.1}$$

Here D is the coefficient of molecular diffusion in free interstitial space, λ is the convective diffusion parameter, δ is the macrodispersion parameter, and v is the filtration velocity, defined as the weighted mean of the filtration velocities in each stratum or the total filtration rate of flow divided by the thickness of the medium under consideration.

In essentially inhomogeneous media, when the filtration velocities are considerable, the first two terms of (0.1) are small by comparison with the third term, i.e., the total dispersion coefficient in this case is [2, 3]

$$D^* \approx \delta v^2 \tag{0.2}$$

<u>1. Formulation of the Problem.</u> We shall consider the motion of a mixture of relative excess concentration C in an closed isotropic porous medium of many strata when there is either infusion or pumping out through a hole of radius r_0, assuming that the filtration flow is stationary and stabilized, and subject to the linear Darcy law. In this case the axisymmetric macrodispersion equation has the form

$$n \frac{\partial C}{\partial t_\pm} + \frac{Q_\pm}{2\pi mr} \frac{\partial C}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(D_\pm^* r \frac{\partial C}{\partial r} \right), \quad D_\pm^* = \delta \left(\frac{Q_\pm}{2\pi mr} \right)^2 \tag{1.1}$$

Here n, m are the weighted mean values of the active porosity and the thickness of the multistratum medium; Q is the flow rate of the filtration of the incoming or outgoing mixture through the hole; r is the distance from the axis of the hole; t is the time. The subscript "+" refers to infusion of the mixture, and the subscript "−" to pumping out. In Eq. (1.1) the mixture flow rate is positive for infusion and negative for pumping out.

We introduce the generalized independent variables

$$x = \pi \frac{r^2 - r_0^2}{m^2}, \quad \tau_\pm = \frac{t_\pm}{\delta n} \tag{1.2}$$

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Then Eq. (1.1) can be written as

$$\frac{\partial^2 C}{\partial x^2} - a_\pm \frac{\partial C}{\partial x} - a_\pm{}^2 \frac{\partial C}{\partial \tau_\pm} = 0, \qquad a_\pm = \frac{m^3}{\delta Q_\pm} \tag{1.3}$$

Equation (1.3) is the one-dimensional diffusion equation with convective term in the region of the variable x defined by (1.2). Hence, subject to the above assumptions, axisymmetric macrodispersion problems can be reduced to one-dimensional problems.

We consider the macrodispersion problem for pumping out through a hole at a flow rate of Q_- following the previous infusion of a mixture during time t_+ through the same hole at a rate Q_+.

In this case the boundary conditions have the form

$$C(x, 0; \tau_+) = \Phi(a_+x, \tau_+), \quad C(\infty, \tau_-; \tau_+) = 0, \quad \left.\frac{\partial C}{\partial x}\right|_{x=0} = 0 \tag{1.4}$$

The initial condition [the first boundary condition of (1.4)] is determined by the distribution of the concentrations of the mixture at the end of its infusion through the hole. This distribution can be determined from the following considerations.

If the infusion is at constant concentration through the hole (boundary condition of the first kind), i.e., if

$$C(0, \tau_+) = 1, C(x, 0) = 0, C(\infty, \tau_+) = 0 \tag{1.5}$$

the initial condition is [4]

$$C(x, 0; \tau_+) = \Phi_1(a_+x, \tau_+) = 0.5\left\{\operatorname{erfc}\left(\frac{a_+x - \tau_+}{2\sqrt{\tau_+}}\right) + \exp(a_+x)\operatorname{erfc}\left(\frac{a_+x + \tau_+}{2\sqrt{\tau_+}}\right)\right\} \tag{1.6}$$

However, as shown in [5-7], a more correct boundary condition at the hole when there is infusion is the boundary condition of the third kind

$$-D_+{}^* \frac{2\pi m r_0}{Q_+} \left.\frac{\partial C}{\partial r}\right|_{r=r_0} + C(r_0, t_+) = 1 \tag{1.7}$$

which in the generalized variables (1.2) is

$$-\frac{1}{a_+} \left.\frac{\partial C}{\partial x}\right|_{x=0} + C(0, \tau_+) = 1 \tag{1.8}$$

In this case the initial condition can be written as [5, 7]

$$C(x, 0; \tau_+) = \Phi_2(a_+x, \tau_+) = 0.5\left\{\operatorname{erfc}\left(\frac{a_+x - \tau_+}{2\sqrt{\tau_+}}\right)\right.$$
$$\left.- \exp(a_+x)\operatorname{erfc}\left(\frac{a_+x + \tau_+}{2\sqrt{\tau_+}}\right) + 2\sqrt{\tau_+}\exp(a_+x)\, i\operatorname{erfc}\left(\frac{a_+x + \tau_+}{2\sqrt{\tau_+}}\right)\right\}. \tag{1.9}$$

2. Solution of the Problem. We transform (1.3) for pumping out, taking account of the initial condition [the first boundary condition of (1.4)], by taking the Laplace transform with respect to the variable τ_-:

$$\frac{d^2W}{dx^2} + a_- \frac{dW}{dx} - a_-{}^2 q\Phi(a_+x, \tau_+) - a_-^2 q W = 0 \tag{2.1}$$

Here

$$W(x, q; \tau_+) = L\{C(x, \tau_-; \tau_+\} = q\int_0^\infty e^{-q\tau_-} C(x, \tau_-; \tau_+)\, d\tau_-$$

according to Carson and Heaviside.

The general integral of (2.1) can be put in the form

$$W(x,\,q;\,\tau_+) = A\exp\left[-\left(\tfrac{1}{2}+\sqrt{q+\tfrac{1}{4}}\right)a_- x\right] + B\exp\left[-\left(\tfrac{1}{2}-\sqrt{q+\tfrac{1}{4}}\right)a_- x\right]$$

$$+\frac{a_- q}{2\sqrt{q+{}^1/_4}}\int_0^x\left\{\exp\left[-\left(\frac{1}{2}+\sqrt{q+\frac{1}{4}}\right)a_-(x-\theta)\right]\right.$$

$$\left.-\exp\left[-\left(\frac{1}{2}-\sqrt{q+\frac{1}{4}}\right)a_-(x-\theta)\right]\right\}\Phi(a_+\theta,\,\tau_+)\,d\theta \tag{2.2}$$

Using the boundary condition at the hole (1.4), we can write (2.2) as follows:

$$W(x,\,q;\,\tau_+) = B\left\{\exp\left[-\left(\frac{1}{2}-\sqrt{q+\frac{1}{4}}\right)a_- x\right] - \frac{{}^1/_2-\sqrt{q+{}^1/_4}}{{}^1/_2+\sqrt{q+{}^1/_4}}\exp\left[-\left(\frac{1}{2}+\sqrt{q+\frac{1}{4}}\right)a_- x\right]\right\}$$

$$+\frac{a_- q}{2\sqrt{q+{}^1/_4}}\int_0^x\left\{\exp\left[-\left(\frac{1}{2}+\sqrt{q+\frac{1}{4}}\right)a_-(x-\theta)\right]\right.$$

$$\left.-\exp\left[-\left(\frac{1}{2}-\sqrt{q+\frac{1}{4}}\right)a_-(x-\theta)\right]\right\}\Phi(a_+\theta,\,\tau_+)\,d\theta \tag{2.3}$$

To determine the constant of integration in (2.3) we use the boundary condition at infinity (1.4). Then

$$B = \frac{a_- q}{2\sqrt{q+{}^1/_4}}\int_0^\infty \exp\left[\left(\frac{1}{2}\sqrt{q+\frac{1}{4}}\right)a_-\theta\right]\Phi(a_+\theta,\,\tau_+)\,d\theta \tag{2.4}$$

From this,

$$W(x,\,q;\,\tau_+) = \frac{a_- q}{2\sqrt{q+{}^1/_4}}\left\{\int_x^\infty \exp\left[\left(\frac{1}{2}-\sqrt{q+\frac{1}{4}}\right)a_-(\theta-x)\right]\Phi(a_+\theta,\tau_+)\,d\theta\right.$$

$$-\frac{{}^1/_2-\sqrt{q+{}^1/_4}}{{}^1/_2+\sqrt{q+{}^1/_4}}e^{-a_- x}\int_0^\infty \exp\left[\left(\frac{1}{2}-\sqrt{q+\frac{1}{4}}\right)a_-(\theta-x)\right]\Phi(a_+\theta,\,\tau_+)\,d\theta$$

$$\left.+\int_0^x \exp\left[\left(\frac{1}{2}+\sqrt{q+\frac{1}{4}}\right)a_-(\theta-x)\right]\Phi(a_+\theta,\tau_+)\,d\theta\right\} \tag{2.5}$$

For x=0, (2.5) has the form

$$W(0,\,q;\,\tau_+) = \frac{a_- q}{{}^1/_2+\sqrt{q+{}^1/_4}}\int_0^\infty \exp\left[\left(\frac{1}{2}-\sqrt{q+\frac{1}{4}}\right)a_-\theta\right]\Phi(a_+\theta,\,\tau_+)\,d\theta \tag{2.6}$$

The original of (2.6) is

$$C(0,\,\tau_-;\,\tau_+) = \int_0^\infty F(\theta,\,\tau_-)\,\Phi(\alpha\theta,\,\tau_+)\,d\theta,$$

$$F(\theta,\,\tau_-) = \frac{1}{\sqrt{\pi\tau_-}}\exp\left[-\frac{(\theta-\tau_-)^2}{4\tau_-}\right] - 0.5e^\theta\,\mathrm{erfc}\left(\frac{\theta+\tau_-}{2\sqrt{\tau_-}}\right),\quad \alpha = a_+/a_- \tag{2.7}$$

Similarly, we can construct the original of (2.5).

The function Φ ($\alpha\,\theta$, τ_+) in (2.7) is defined by the initial conditions (1.6) or (1.9), depending on the boundary conditions for infusion of the mixture through the hole.

3. The Asymptotic Representation of the Solution for a Boundary Condition of the First Kind. We consider first the asymptotic representation of the solution for x=0. To this end we transform Eq. (2.6) with initial condition (1.6) by taking the Laplace transform with respect to τ_+:

$$Y(0,\,q;\,p) = \frac{a_-\left({}^1/_2-\sqrt{q+{}^1/_4}\right)}{a_-\left({}^1/_2-\sqrt{q+{}^1/_4}\right)+a_+\left({}^1/_2-\sqrt{p+{}^1/_4}\right)} \tag{3.1}$$

The asymptotic representation of (3.1) and its original for large values of τ_+ have the forms

$$Y(0, q; p) \approx \frac{a_-(1/2 - \sqrt{q + 1/4})}{a_-(1/2 - \sqrt{q+1/4}) - a_+p}, \quad C(0, \tau_-; \tau_+) \approx 1 - \Phi_1(\tau_+/\alpha, \tau_-) \tag{3.2}$$

The asymptotic representations of (3.2) for large values of τ_- and for $\tau_- > \alpha\tau_+$ have the forms

$$Y(0, q; p) \approx -\frac{a_-q}{a_-q + a_+p(q+1)}, \quad C(0, \tau_-; \tau_+) \approx 1 - J(\tau_+/\alpha, \tau_-) \tag{3.3}$$

respectively, and coincide with (3.2) with an error of not more than 5% if [3]

$$\tau_- > \tau_+/\alpha > 0.1 \tag{3.4}$$

In (3.3)

$$J(\sigma, \chi) = 1 - e^{-\chi}\int_0^\sigma e^{-z} I_0(2\sqrt{\chi z})\,dz \tag{3.5}$$

where J is the function introduced by Schumann [8], and I_0 is the modified Bessel function of the second kind and zero order.

Now we consider the asymptotic representation of (3.1) and its original for large values of τ_-:

$$Y(0, q; p) \approx \frac{a_-q}{a_-q - a_+(1/2 - \sqrt{p + 1/4})}, \quad C(0, \tau_-; \tau_+) \approx \Phi_1(\alpha\tau_-, \tau_+) \tag{3.6}$$

The asymptotic representation of (3.6) and its original for large values of τ_+ and $\tau_+ > \alpha\tau_-$ are

$$Y(0, q; p) \approx \frac{a_-q(p+1)}{a_-q(p+1) + a_+p}, \quad C(0, \tau; \tau_+) \approx J(\alpha\tau_-, \tau_+) \tag{3.7}$$

respectively.

The expansion (3.7) coincides with (3.6) with an error of not more than 5% if [3]

$$\tau_+ > \alpha\tau_- > 0.1 \tag{3.8}$$

To estimate the applicability of the above asymptotic expansions (3.2) and (3.6), we write the solution (2.7) as

$$C(0, \tau_-; \tau_+) \approx \int_0^\beta F(\theta, \tau_-)\,d\theta + \int_\beta^\gamma F(\theta, \tau_-)\Phi_1(\alpha\theta, \tau_+)\,d\theta \tag{3.9}$$

defining the limits of integration β and γ as the boundaries of the region of dispersion formed on infusion, with an error of not more than 0.1% in (3.9):

$$\beta = \alpha^{-1}(\tau_+ - 5\sqrt{\tau_+}); \quad \gamma = \alpha^{-1}(\tau_+ + 5\sqrt{\tau_+}) \tag{3.10}$$

In the region $\beta \le \theta \le \gamma$ the function $\Phi_1(\alpha\theta, \tau_+)$ varies in practice from 1 to 0. If we assume that in this region

$$\Phi_1(\alpha\theta, \tau_+) \equiv 1 \tag{3.11}$$

then

$$C(0, \tau_-; \tau_+) < 1 - \Phi_1([\tau_+ + 5\sqrt{\tau_+}]\alpha^{-1}, \tau_-) \tag{3.12}$$

A comparison of (3.2) and (3.12) determines the error in using (3.2):

$$\Delta < \frac{|\Phi_1([\tau_+ + 5\sqrt{\tau_+}]\alpha^{-1}, \tau_-) - \Phi_1(\tau_+/\alpha, \tau_-)|}{1 - \Phi_1(\tau_+/\alpha, \tau_-)} \tag{3.13}$$

Similarly, defining the limits of integration β and γ in (3.9) as the boundaries of the region of dispersion for pumping out, we obtain the error in using (3.6) in the form

$$\Delta < 1 - \frac{\Phi_1(\alpha[\tau_- + 5\sqrt{\tau_-}], \tau_+]}{\Phi_1(\alpha\tau_-, \tau_+)} \tag{3.14}$$

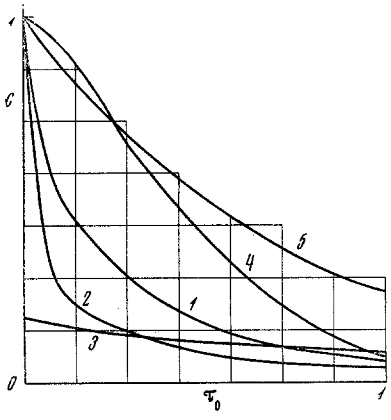

Fig. 1

In addition, we can construct asymptotic representations of the solution for large values of x. It can be shown that for large values of τ_+ the asymptotic form of the solution, the representation of which is defined by (2.5), has the form

$$C(x,\tau_-;\tau_+)\approx 1-0.5\left\{\operatorname{erfc}\left(\frac{a_-[\tau_+/a_+-x]-\tau_-}{2\sqrt{\tau_-}}\right)+\exp(\tau_+/\alpha)\operatorname{erfc}\left(\frac{a_-[\tau_+/a_++x]+\tau_-}{2\sqrt{\tau_-}}\right)\right\} \tag{3.15}$$

where $\tau_+ \gg a_+x$, and the asymptotic representation of the solution for large values of τ_- is defined as follows:

$$C(x,\tau_-;\tau_+)\approx \Phi_1(a_+[\tau_-/a_- - x],\ \tau_+) \tag{3.16}$$

The conditions for (3.15) and (3.16) to be used can be constructed by analogy with the conditions (3.13) and (3.14).

4. Asymptotic Representations of the Solution for a Boundary Condition of the Third Kind. Asymptotic expansions of the solution for x=0, with initial condition (1.9), can be obtained from the following considerations. We consider the double Laplace transform of (2.7)

$$Y(0,q;p)=\frac{a_-(^1/_2-\sqrt{q+^1/_4})(^1/_2-\sqrt{p+^1/_4})}{p[a_-(^1/_2-\sqrt{q+^1/_4})+a_+(^1/_2-\sqrt{p+^1/_4})]} \tag{4.1}$$

For large values of τ_+ (4.1) becomes

$$Y(0,q;p)\approx\frac{a_-(^1/_2-\sqrt{q+^1/_4})}{(p+1)[a_-(^1/_2-\sqrt{q+^1/_4})-a_+p]} \tag{4.2}$$

The inverse transformation of (4.2) with respect to p is

$$W(0,q;\tau_+)\approx 1-\frac{\alpha(^1/_2-\sqrt{q+^1/_4})}{^1/_2-\sqrt{q+^1/_4}+\alpha}e^{-\tau_+}-\frac{\alpha}{^1/_2-\sqrt{q+^1/_4}+\alpha}\exp\left[\left(\frac{1}{2}-\sqrt{q+^1/_4}\right)\frac{\tau_+}{\alpha}\right] \tag{4.3}$$

This corresponds to the asymptotic form of the original:

$$\begin{aligned}C(0,\tau_-;\tau_+)\approx 1&-\frac{1}{2(\alpha+1)}e^{-\tau_+}\{\operatorname{erfc}(^1/_2\sqrt{\tau_-})+(\alpha+^1/_2)\exp[\alpha(\alpha+1)\tau_-]\operatorname{erfc}(-[\alpha+^1/_2]\sqrt{\tau_-})\}\\&-0.5\operatorname{erfc}\left(\frac{\tau_+/\alpha-\tau_-}{2\sqrt{\tau_-}}\right)-\frac{\alpha}{2(\alpha-1)}\exp(\tau_+/\alpha)\operatorname{erfc}\left(\frac{\tau_+/\alpha+\tau_-}{2\sqrt{\tau_-}}\right)\\&+\exp[-\tau_++\alpha(\alpha+1)\tau_-]\operatorname{erfc}\left(\frac{\tau_+/\alpha-[2\alpha+1]\tau_-}{2\sqrt{\tau_-}}\right)\end{aligned} \tag{4.4}$$

the asymptotic form of which for large τ_+, in turn, is

$$C(0,\tau_-;\tau_+)\approx 1-\Phi_1(\tau_+/\alpha,\ \tau_-) \tag{4.5}$$

i.e., it coincides with the asymptotic representation (3.2) of the solution for a boundary condition of the first kind.

For large τ_- the asymptotic form of the representation (4.1) and its inverse transformation with respect to q and the original, respectively, are

$$Y(0,q;p)=\frac{a_-q(^1/_2-\sqrt{p+^1/_4}}{p[a_-q-a_+(^1/_2-\sqrt{p+^1/_4})]} \tag{4.6}$$

$$U(0,\tau_-;p)\approx-\frac{^1/_2-\sqrt{p+^1/_4}}{p}\exp\left[\left(\frac{1}{2}-\sqrt{p+\frac{1}{4}}\right)\alpha\tau_-\right]$$

$$C(0,\tau_-;\tau_+)\approx\Phi_2(\alpha\tau_-;\ \tau_+)$$

As we know [7, 9], the asymptotic representation of (1.9) coincides with (1.6). Hence, the asymptotic forms of the solutions (4.6) and (3.6) coincide for large τ_- and large τ_+.

The conditions for these asymptotic expansions and also the asymptotic representations for any x to be applicable can be constructed in the same way as was done in § 3.

5. Analysis of the Results. A Numerical Example. The exact solution of the problem corresponding to (2.5) is very complicated, but in particular conditions it is sufficient to use the asymptotic expansions.

Indeed, if infusion is at a rate much less than the pumping-out rate, the dispersion which arose in infusion can be neglected, assuming that transport is determined for infusion only by the forced convection. In this case, Eq. (13) has to be solved with the boundary conditions

$$\begin{gathered} C(x,0,\tau_+)=\begin{cases}1 & \text{for } x\leqslant \tau_+/a_+\\ 0 & \text{for } x>\tau_+/a_+\end{cases}\\ \left.\frac{\partial C}{\partial x}\right|_{x=0}=0\\ C(\infty,\tau_-,\tau_+)=0, \end{gathered} \tag{5.1}$$

The solution of the problem in this formulation can be deduced from the solution given in [5]. This solution is very complicated. But, following [9], we can show that in particular conditions the solution virtually coincides with (3.2) and (3.15).

Similarly, in the case when the pumping-out flow rate is much less than the infusion flow rate we can neglect dispersion arising during pumping out, assuming that its effect on the dispersion arising during infusion is small. In this case, in Eq. (1.3) we omit the diffusion term,and its solution coincides with (3.6), (3.16), and (4.6).

We now consider a numerical example. We put $\tau_+=0.4$, $a_+=2$, $a_-=1$. Curves for the relative concentration in the hole through which the mixture is pumped out are given in Fig. 1. In this case we assume that a boundary condition of the first kind holds for infusion, i.e., we use the initial condition (1.6). Curve 1 corresponds to the exact solution of the problem (2.7), curve 2 to the asymptotic expansion (3.2), curve 3 to the asymptotic form (3.3), curve 4 to the asymptotic representation (3.6), and curve 5 to (3.7).

From Fig. 1, in spite of the fact that the flow rate of pumping out is twice that of infusion, asymptotic representation (3.2) quite accurately reflects the nature of the process for small values of the pumping time.

The results obtained in this paper make it possible to lay the basis for the conduct of tests on water-bearing tables in order to determine the parameters of transport by the single-hole method. A similar formulation of a field experiment is very urgent for the study of deep water-bearing tables.

As analysis of the asymptotic representations of the solutions shows, the simplest experiment is one in which the infusion flow rate either is much greater or is much less than the pumping flow rate.

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